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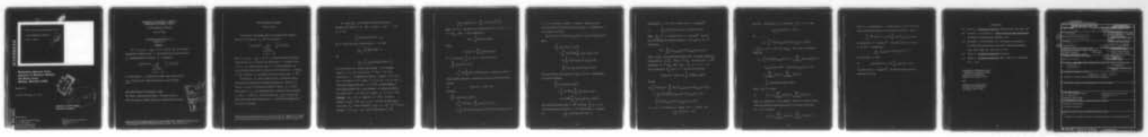

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ON AN EXTREMAL PROBLEM

Paul G. Nevai

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ON AN EXTREMAL PROBLEM

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ABSTRACT

Let $X = (x_1, x_2, \ldots, x_N)$, $f : \mathbb{R} \to \mathbb{C}$ and let $\mathbb{P}_n$ be the class of polynomials of degree at most n. The generalized Christoffel function Λ_n corresponding to the measure $d\alpha$ is defined by

$$\Lambda_n(X; f, N, d\alpha) = \min_{\substack{\pi \in \mathbb{P}_{n-1} \\ \pi(x_i) = f(x_i) \\ i = 1, 2, \ldots, N}} \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t) .$$

It is shown that if α satisfies some rather weak conditions then $\lim_{n \to \infty} n \Lambda_n(X; f, N, d\alpha)$ exists and the limit is also evaluated.

AMS (MOS) Subject Classification: 42A52

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ON AN EXTREMAL PROBLEM

Paul G. Nevai

The purpose of the present note is to investigate the asymptotic behavior of the functions $\Lambda_n : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\Lambda_n(X; f, N, d\alpha) = \min_{\substack{\pi \in \mathbb{P}_{n-1} \\ \pi(x_i) = f(x_i) \\ i = 1, 2, \ldots, N}} \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t) .$$

Here $X = (x_1, x_2, \ldots, x_N)$, $f : \mathbb{R} \to \mathbb{C}$ is a fixed and almost everywhere finite function, $\mathbb{P}_n$ is the set of all polynomials π of degree at most n and α is a weight function, that is α is nondecreasing on $\mathbb{R}$, it has infinitely many points of increase and every polynomial π belongs to $L^2_{d\alpha}$. Therefore Λ_n is defined and finite for almost every $X \in \mathbb{R}^N$.

Estimates for Λ_n lead to several results in probability theory, statistics and in the theory of orthogonal polynomials. (See [1], [2] and [4].) In fact, it is not hard to explicitly compute Λ_n ([3]) but the formula for Λ_n is so complicated that it cannot be used to estimate Λ_n when α is not nice. *It will be shown that* $\lim_{n \to \infty} n\Lambda_n(X; f, N, d\alpha)$ exists under rather weak assumptions on α and the corresponding limit will also be calculated.

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Let $\{p_n(d\alpha)\}_{n=0}^{\infty}$ be the system of polynomials which is orthogonal with respect to $d\alpha$, that is $p_n(d\alpha, x) = \gamma_n x^n + \cdots$ with $\gamma_n > 0$ and

$$\int_{-\infty}^{\infty} p_n(d\alpha, t) p_m(d\alpha, t) d\alpha(t) = \delta_{nm} .$$

Let M denote the class of those weights α for which

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} t p_n^2(d\alpha, t) d\alpha(t) = 0$$

and

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} t p_{n-1}(d\alpha, t) p_n(d\alpha, t) d\alpha(t) = \frac{1}{2} .$$

Let us remark that M contains many weights. If, for instance, $\mathrm{supp}(d\alpha) = [-1, 1]$ and $\log \alpha'(\cos\theta) \in L^1$ then $\alpha \in M$ ([5]). Furthermore, if α is absolutely continuous, $\mathrm{supp}(d\alpha) = [-1, 1]$ and $\alpha'(t) = \varphi(t) \exp\{-(1 - t^2)^{-\frac{1}{2}}\}$ where $0 < c \le \varphi(t)$ for $-1 \le t \le 1$ and φ is Riemann integrable then also $\alpha \in M$ ([4]). Another example is the Pollaczek weight ([5]). In the following Δ denotes an interval. It is known that if $\alpha \in M$ then $\Delta \subset \mathrm{supp}(d\alpha)$ iff $\Delta \subset [-1, 1]$ ([4]).

<u>THEOREM</u>. Let $\alpha \in M$, $\Delta \subset \mathrm{supp}(d\alpha)$, $1/\alpha' \in L^1(\Delta)$. Let exist a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ with $\varepsilon_k \ge 0$, $\varepsilon_k \searrow 0$ such that for every fixed k the function Ψ_k, which is defined by $\Psi_k(t) = [(1 - \varepsilon_k)^2 - t^2]^{-\frac{1}{2}} \log \alpha'(t)$, belongs to $L^1(-1 + \varepsilon_k, 1 - \varepsilon_k)$. Then for almost every $X \in \Delta^N$

$$\lim_{n\to\infty} n\Lambda_n(X;f,N,d\alpha) = \pi \sum_{i=1}^{N} |f(x_i)|^2 \alpha'(x_i)\sqrt{1-x_i^2} .$$

Proof. For $N = 1$ the Theorem has been proved in [4]. Now let $N > 1$.

If $\pi \in \mathbb{P}_{n-1}$ then π can be expressed as

$$\pi(x) = \int_{-\infty}^{\infty} \pi(t)K_n(d\alpha, x, t)d\alpha(t)$$

where

$$K_n(d\alpha, x, t) = \sum_{k=0}^{n-1} p_k(d\alpha, x)p_k(d\alpha, t) .$$

If X is given and $\prod_{i<j} (x_i - x_j) \neq 0$ then we can write

$$\sum_{i=1}^{N} \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) =$$

$$= \int_{-\infty}^{\infty} \pi(t)[\sum_{i=1}^{N} \operatorname{sign} \pi(x_i)^2 \pi(x_i)\Lambda_n(x_i, 1, 1, d\alpha)K_n(d\alpha, x_i, t)]d\alpha(t) .$$

Using Cauchy's inequality, orthogonality relations and the well known fact that

(1) $$K_n(d\alpha, x, x) = \Lambda_n(x;1, 1, d\alpha)$$

we obtain

$$[\sum_{i=1}^{N} \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha)]^2 \leq$$

$$\leq \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t) \cdot [\sum_{i=1}^{N} \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) +$$

$$+ 2 \sum_{j<i} |\pi(x_i)||\pi(x_j)|\Lambda_n(x_i;1, 1, d\alpha)\Lambda_n(x_j;1, 1, d\alpha)|K_n(d\alpha, x_i, x_j)|] .$$

If $\alpha \in M$ then clearly $\operatorname{supp}(d\alpha)$ is compact. Therefore by the Christoffel-Darboux formula there exists a number C depending on X and $\operatorname{supp}(d\alpha)$ such that

$$|K_n(d\alpha, x_i, x_j)| \le C[\,|p_n(d\alpha, x_i)p_{n-1}(d\alpha, x_j)| + |p_{n-1}(d\alpha, x_i)p_n(d\alpha, x_j)|\,] .$$

Hence

$$[\sum_{i=1}^{N} \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha)]^2 \le$$

$$\le \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t)[\sum_{i=1}^{N} \Lambda_n(x_i, |\pi(x_i)|, 1, d\alpha) +$$

$$+ 4C \sum_{i=1}^{N} |\pi(x_i)|\,|p_n(d\alpha, x_i)|\,\Lambda_n(x_i; 1, 1, d\alpha) \cdot$$

$$\cdot \sum_{i=1}^{N} |\pi(x_i)|\,|p_{n-1}(d\alpha, x_i)|\,\Lambda_n(x_i; 1, 1, d\alpha)] .$$

Formula (1) implies that $p_{n-1}^2(d\alpha, x_i)\, \Lambda_n(x_i, 1, 1, d\alpha) \le 1$. Consequently by Cauchy's inequality

$$[\sum_{i=1}^{N} \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha)]^2 \le$$

$$\le \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t) \sum_{i=1}^{N} \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) \cdot$$

$$\cdot \{1 + 4C\sqrt{N}\, [\sum_{i=1}^{N} \Lambda_n(x_i; |p_n(d\alpha, x_i)|, 1, d\alpha)]^{\frac{1}{2}}\} .$$

This inequality holds for each $X \in \mathbb{R}^N$ such that $\prod_{i<j} (x_i - x_j) \neq 0$.

It has been proved in [4] that if $\alpha \in M$ then for every $x \in \operatorname{supp}(d\alpha)$

(2) $$\lim_{n \to \infty} \Lambda_n(x; |p_n(d\alpha, x)|, 1, d\alpha) = 0 .$$

Consequently if $\alpha \in M$ then for almost every $X \in [\operatorname{supp}(d\alpha)]^N$

$$(3) \qquad \sum_{i=1}^{N} \Lambda_n(x_i; |f(x_i)|, 1, d\alpha) \leq [1 + \sigma(1)]\Lambda_n(X, f, N, d\alpha)$$

where $\lim_{n\to\infty} \sigma(1) = 0$ for almost every $X \in [\operatorname{supp}(d\alpha)]^N$. Our next aim is to establish the converse inequality. For a given X let π^X be defined by

$$\pi^X(t) = \sum_{i=1}^{N} f(x_i)\Lambda_{n-N+1}(x_i;1, 1, d\alpha)K_{n-N+1}(d\alpha, t, x_i)\ell_i(t)$$

where $\{\ell_i\}_{i=1}^N$ are the fundamental polynomials of Lagrange interpolation corresponding to $\{x_i\}_{i=1}^N$. Clearly $\pi^X \in \mathbb{P}_{n-1}$ for almost every $X \in \mathbb{R}^N$. Furthermore, if $\pi^X \in \mathbb{P}_{n-1}$ then $\pi^X(x_i) = f(x_i)$ for $i = 1, 2, \ldots, N$. Let us compute the $L^2_{d\alpha}$ norm of π^X. Introducing the notation

$$G_n(d\alpha, g, x) = \Lambda_n(x;1, 1, d\alpha) \int_{-\infty}^{\infty} g(t)K_n^2(d\alpha, x, t)d\alpha(t)$$

we have

$$(4) \quad \int_{-\infty}^{\infty} |\pi^X(t)|^2 d\alpha(t) = \sum_{i=1}^{N} |f(x_i)|^2 \Lambda_{n-N+1}(x_i;1, 1, d\alpha)G_{n-N+1}(d\alpha, \ell_i^2, x_i)$$

$$+ 2 \sum_{i<j} \operatorname{Re}[f(x_i)\overline{f(x_j)}]\Lambda_{n-N+1}(x_i;1, 1, d\alpha)\Lambda_{n-N+1}(x_j;1, 1, d\alpha) \cdot$$

$$\cdot \int_{-\infty}^{\infty} \ell_i(t)\ell_j(t)K_{n-N+1}(d\alpha, t, x_i)K_{n-N+1}(d\alpha, t, x_j)d\alpha(t) .$$

If $\alpha \in M$, g is continuous on $\operatorname{supp}(d\alpha)$ and $x \in \operatorname{supp}(d\alpha)$ then

$$\lim_{n\to\infty} G_n(d\alpha, g, x) = g(x)$$

(see [4]), in particular, if X is such that $\prod_{i<j} (x_i - x_j) \neq 0$ then

$$\lim_{n\to\infty} G_n(d\alpha, \ell_i^2, x_i) = \ell_i^2(x_i) = 1 .$$

Let

$$I_{ij} = \int_{-\infty}^{\infty} \ell_i(t)\ell_j(t)K_{n-N+1}(d\alpha, t, x_i)K_{n-N+1}(d\alpha, t, x_j)d\alpha(t) .$$

If $\prod_{k<\ell} (x_k - x_\ell) \neq 0$ then $\ell_i\ell_j \in \mathbb{P}_{2N-2}$. By a direct calculation we obtain that

$$I_{ij} = \int_{-\infty}^{\infty} \ell_i(t)\ell_j(t)[K_{n-3N+3}(d\alpha, x_i, t) + \sum_{\ell=n-3N+3}^{n-N} p_k(d\alpha, x_i)p_k(d\alpha, t)] \cdot$$

$$\cdot K_{n-N+1}(d\alpha, x_j, t)\, d\alpha(t) = \ell_i(x_j)\ell_j(x_j)K_{n-3N+3}(d\alpha, x_i, x_j) +$$

$$+ \sum_{k=n-3N+3}^{n-N} p_k(d\alpha, x_i) \sum_{\ell=k-2N+2}^{n-N} p_\ell(d\alpha, x_j) \cdot$$

$$\cdot \int_{-\infty}^{\infty} \ell_i(t)\ell_j(t)p_k(d\alpha, t)p_\ell(d\alpha, t)d\alpha(t) .$$

Since $\ell_i(x_j) = 0$ we get

$$|I_{ij}| \leq C_1 \sum_{k=n-5N}^{n-N} |p_k(d\alpha, x_i)| \sum_{\ell=n-5N}^{n-N} |p_\ell(d\alpha, x_j)| \tag{5}$$

where C_1 depends on X and $\mathrm{supp}(d\alpha)$. Using the recurrence formula which the orthogonal polynomials satisfy it can easily be seen that for $\alpha \in M$ (5) implies

$$|I_{ij}| \leq C_2 \sum_{k=n-N-1}^{n-N} |p_k(d\alpha, x_i)| \sum_{\ell=n-N-1}^{n-N} |p_\ell(d\alpha, x_j)|$$

where C_2 is independent of n. Applying now (1) and (2) the second sum on the right hand side of (4) can easily be estimated. We get

$$\int_{-\infty}^{\infty} |\pi^X(t)|^2 d\alpha(t) = [1 + \sigma(1)] \sum_{i=1}^{N} |f(x_i)|^2 \Lambda_{n-N+1}(x_i;1,1,d\alpha)$$

for almost every $X \in [\mathrm{supp}(d\alpha)]^N$. It has been proved in [4] that if $\alpha \in M$ and $x \in \mathrm{supp}(d\alpha)$ then

$$\lim_{n\to\infty} \Lambda_{n-N+1}(x;1,1,d\alpha)\Lambda_n(x;1,1,d\alpha)^{-1} = 1$$

for every fixed N. Thus

$$\Lambda_n(X;f,N,d\alpha) \le [1 + \sigma(1)] \sum_{i=1}^{N} \Lambda_n(x_i;|f(x_i)|,1,d\alpha) \tag{6}$$

for almost every $X \in [\mathrm{supp}(d\alpha)]^N$. The Theorem follows now from estimates (3) and (6).

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Asymptotic estimate is given for a generalization of Christoffel functions.

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